ISRAEL JOURNAL OF MATHEMATICS 114 (1999), 189–203

ARNON COMPLETION OF THE DYER–LASHOF ALGEBRA*

BY

H. E. A. CAMPBELL

Department of Mathematics and Statistics, Queen's University
Kingston, Ontario, K7L 3N6, Canada
e-mail: campbelh@queensu.ca

AND

N. E. KECHAGIAS

Department of Mathematics, University of Ioannina
Ioannina, 45110, Greece
e-mail: nkechag@cc.uoi.gr

ABSTRACT

In his PhD thesis, Arnon [1] builds a completion of the Dickson algebras which contains a "free root" algebra D_{fin} on the top Dickson classes. Hu'ng [5] has shown that this algebra is in fact isomorphic to a similar completion $(A_\mu)^*$ of the dual of the Steenrod algebra A^*. Arnon also completed the Steenrod algebra A with respect to its halving homomorphism to obtain A_μ. Here we study an analogous completion of the Dyer–Lashof algebra R to obtain R_μ with canonical subcoalgebras $R_\mu[n]$. Unlike the Steenrod algebra, we may further complete R_μ with respect to length to obtain $\widehat{R_\mu}$. It turns out, somewhat surprisingly, that the dual $\left(\hat{R}_\mu\right)^*$ contains $(A_\mu)^*$ as a dense subalgebra.

* This research is supported in part by the Natural Sciences and Engineering Research Council of Canada.
Received May 15, 1997

1. Introduction

In this paper we study some interesting ideas from D. Arnon's MIT PhD thesis [1] written under the supervision of M. Hopkins. Arnon defines two versions of completion. He completes the Steenrod algebra A with respect to its halving homomorphism to obtain A_μ and studies the process of attaching square roots to each polynomial in the Dickson algebras D_n in n variables ([4]) over $\mathbf{F}_2$. The resulting algebra $D_n^{\surd}$ is a free root algebra on Dickson's generators. Once this is done, the resulting Dickson algebras form an inverse system, and the limit $D\surd$ contains the free root algebra on the top Dickson classes, here denoted $D_{fin}^{\surd}$. Hu'ng [5] shows that the completion $(A_\mu)^*$ of A^* (the graded hom-dual of A, [8]) with respect to its squaring map is isomorphic to $D_{fin}^{\surd}$. We observe that $D_{fin}^{\surd}$ carries a natural coproduct and the isomorphism described by Hu'ng is in fact an isomorphism of Hopf algebras. As Hu'ng notes, the halving homomorphism in A and the squaring map in A^* are dual but their completions are not of finite type, so we cannot deduce that $\hom_{\mathbf{F}_2}(A_\mu, \mathbf{F}_2) \not\cong A_\mu^*$.

The Dyer–Lashof algebra R and a related less well-known algebra W are completed with respect to their halving homomorphisms. Two new completed Hopf algebras graded over $\mathbf{N}[\frac{1}{2}]$ are obtained which are denoted R_μ and W_μ. The Dyer–Lashof algebra R has extra structure when compared to A — a decomposition by length into canonical subcoalgebras $R[n]$. It is well-known that $R[n]^* \cong D_n$; see, for example, [3, 7, 10]. We have canonical subcoalgebras $R[n]_\mu$ and $W[n]_\mu$ as well and we obtain $R[n]_\mu^* \cong D_n^{\surd}$. There is a similar result relating $W[n]_\mu^*$ to the ring of upper triangular invariants [2, 9] which is here denoted $H_n^{\surd}$.

Finally, just as Arnon forms an inverse limit of the Dickson algebras, we form a direct limit of the coalgebras $R[n]_\mu$ with respect to a map θ to obtain a coalgebra $\widehat{R_\mu}$. We have (Theorem 4.9) that $\left(\widehat{R_\mu}\right)^* \cong D\surd$ together with a similar result for $\left(\widehat{W_\mu}\right)^*$. Coupled with Hu'ng's result, we may conclude, rather surprisingly, that $\left(\widehat{R_\mu}\right)^*$ contains a subalgebra isomorphic to A_μ^*.

This paper is written over the finite field $\mathbf{F}_2$. The paper is organized as follows. In §2 we collect classical necessary material from the literature and the work of Arnon. We extend his study to the upper triangular case. In §3 we apply Arnon's ideas to the Dyer–Lashof and a related algebra. In the last section, we detail the theorems described above. The paper is more or less self-contained, but we imagine many readers will want to refer to [1] and [5].

ACKNOWLEDGEMENT: We thank F. Peterson for his advice and the editor for making this paper readable. The second author wishes to express his gratitude for hospitality at Queen's University where this work started.

2. Generalized Dickson and certain subalgebras

Let $P_n = \bigoplus_{t\geq 0}(P_n)_t$ denote the graded polynomial algebra $\mathbf{F}_2[y_1,\dots,y_n]$ equipped with the usual actions of $G_n = \mathrm{GL}_n(\mathbf{F}_2)$ and the Steenrod algebra A, where the degree of y_i (here denoted by $|y_i|$) is 1. Two important rings of invariant polynomials are considered. Let $U_n \subset G_n$ be the subgroup consisting of upper triangular matrices and V_i be the vector space with basis $\{y_1,\dots,y_i\}$. Let

$$f_i(t) = \prod_{v\in V_i}(t-v) = \sum_{j=0}^{i} d_{j,i}t^{2^{i-j}}.$$

Observe that the coefficients $d_{j,i}$ are invariant under $\mathrm{Gl}(V_i)$ and that $|d_{j,i}| = 2^i - 2^{i-j}$. Define $h_i = f_{i-1}(y_i)$ and note $|h_i| = 2^{i-1}$.

THEOREM 2.1 ([4, 9, 11]): *$H_n := P_n^{U_n} = \mathbf{F}_2[h_1,\dots,h_n]$ and $D_n := P_n^{G_n} = \mathbf{F}_2[d_{1,n},\dots,d_{n,n}]$.*

The algebra D_n is known as the Dickson algebra.

Arnon considers the Frobenius map $d\colon P_n \to P_n$ given by $d(f) = f^2$ and studies a kind of "mapping telescope" of functions such as d. In the first instance one needs a degree preserving map, so we define a map still called $d\colon P_n \to \frac{1}{2}P_n$ where now $(\frac{1}{2}P_n)_t = (P_n)_{2t}$. In general, let $2^kP_n = \bigoplus (P_n)_{2^{-k}t}$ such that $2^{-k}t \in \mathbf{N}$.

Let us consider $\frac{1}{2}P_n$ and P_n as algebras graded over $\mathbf{Z}[\frac{1}{2}]$ with the latter algebra 0 in fractional and negative degrees. Now d is a degree-preserving map and by iterating the construction just given one extends

$$d\colon \frac{1}{2^{t-1}}P_n \to \frac{1}{2^t}P_n$$

and obtains the direct limit

$$P_n^{\surd} := \varinjlim\left(\frac{1}{2^t}P_n, d\right).$$

We note that the algebra $P_n^{\surd}$ has the property that all elements have square roots. For example, $y_1 \in (1/2^t)P_n$ is the 2^t-root of y_1.

Definition 2.1: A root algebra A is a $\mathbf{Z}[\frac{1}{2}]$-graded commutative algebra over $\mathbf{F}_2$ where the degree preserving homomorphism $d\colon A \to \frac{1}{2}A$ defined by $d(x) = x^2$ is an isomorphism.

PROPOSITION 2.2: *$P_n^{\surd}$ is a* **free root** *algebra on its generators.*

Proof: Let $Y_n = \{y_1, \dots, y_n\}$ and R be a root algebra (R contains more than one element and d: $R \to \frac{1}{2}R$ is an isomorphism) with f: $Y_n \to R$ a map of sets. We show that $\exists!$ F: $P_n^{\surd} \to R$ which extends f. Let $(y_i)_t \in (1/2^t)P_n$. Define $F(y_i)_t = (y_i, t)$ such that $d^t(y_i, t) = ((fy_i))_t$ and extend to an algebra map. For a monomial $(y_1^{i_1} \cdots y_n^{i_n})_t$, we have $F(y_1^{i_1} \cdots y_n^{i_n})_t = r$ such that $d^t(r) = (y_1^{i_1} \cdots y_n^{i_n})_t$. This is well defined because of the compatibility of the squaring map. Suppose $F(y_i) = G(y_i) = f(y_i)$. Then $F(y_i)_t^{2^t} = G(y_i)_t^{2^t} \Rightarrow (F(y_i)_t)^{2^t} = (G(y_i)_t)^{2^t} \Rightarrow d^t F(y_i)_t = d^t G(y_i)_t \Rightarrow F(y_i)_t = G(y_i)_t$. d: $P_n^{\surd} \to \frac{1}{2}P_n^{\surd}$ is an isomorphism. ∎

Arnon goes on to discuss the action of $\mathrm{GL}_n(\mathbf{F}_2)$ as grade-preserving algebra automorphisms of $P_n^{\surd}$. Let

$$D_n^{\surd} := \varinjlim \left(\frac{1}{2^t} D_n, d\right).$$

Definition 2.2: (a) Let

$${}_n\omega_m = \sum_{\substack{s_1+\cdots+s_n=m,\\ s_i=0 \text{ or } 2^{r_i}, r_i\in\mathbf{Z}}} y_1^{s_1} \cdots y_n^{s_n}$$

be the Peterson polynomial of degree m on n variables. Here $m \in \mathbf{N}[\frac{1}{2}]$.

(b) Let

$${}_n o_m = \sum_{\substack{s_1+\cdots+s_n=m, s_n\neq 0\\ s_i=0 \text{ or } 2^{r_i}, r_i\in\mathbf{Z}}} y_1^{s_1} \cdots y_n^{s_n}$$

be the polynomial of degree m on n variables. Here $m \in \mathbf{N}[\frac{1}{2}]$.

LEMMA 2.3:

(i) *The Peterson polynomial ${}_n\omega_m$ is a GL_n-invariant and ${}_n o_m$ is its U_n-invariant analogue.*

(ii) $({}_n\omega_m)^2 = {}_n\omega_{2m}$.

(iii) ${}_n\omega_m = {}_n o_m + {}_{n-1}\omega_m$.

(iv) ${}_n\omega_{2^n-2^i} = d_{n-i,n}$ *and* ${}_n o_{2^n-1} = h_n$.

Proof: (i) The first claim has been proved by Arnon. U_n is generated by elementary matrices $\{(e_{ij}) \mid j \geq i\}$; $e_{ij}({}_n o_m) = {}_n o_m + \sum y_1^{t_1} \cdots y_n^{t_n}$. Here the sum is over the n-tuples $(t_1, \dots, t_n)$ such that $t_j = 0$, $t_i = s_i + s_j$, and $t_q = s_q$ otherwise. The last sum is zero over F_2 because ${}_n o_m$ is symmetric.

(iv) follows from the definitions. ∎

PROPOSITION 2.4 ([1]):

(a) $D_n^{\surd} \equiv \left(P_n^{\surd}\right)^{\mathrm{GL}_n}$, $n < \infty$.

(b) $D_n^{\surd}$ *is the free root algebra generated by* $\{{}_n\omega_{2^i-1} \mid 1 \le i \le n\}$, *where* ${}_n\omega_{2^{n-i}-1}$ *is the image of* $d_{i,n}$ *under the embedding of* D_n *in* $D_n^{\surd}$.

Proof: Let $\Phi\colon \left(P_n^{\surd}\right)^{\mathrm{GL}_n} \to D_n^{\surd}$ be given by $\Phi(\bar f) = j_t(f_t)$. Here $i_t\colon P_n \to P_n^{\surd}$, $j_t\colon D_n \to D_n^{\surd}$ and $i_t(f_t) = \bar f$.

Let

$$\varphi_t\colon \frac{1}{2^t}D_n \to \left(\frac{1}{2^t}P_n\right)^{\mathrm{GL}_n}$$

be the evident map which is an isomorphism. The following diagram is commutative:

$$\begin{array}{ccc} \frac{1}{2^t}D_n & \xrightarrow{d} & \frac{1}{2^{t+1}}D_n \\ \downarrow & & \downarrow \\ \left(\frac{1}{2^t}P_n\right)^{\mathrm{GL}_n} & \to & \left(\frac{1}{2^{t+1}}P_n\right)^{\mathrm{GL}_n} \end{array}$$

Here the vertical maps are φ_t and φ_{t+1}, respectively. It follows now that Φ is an isomorphism. ∎

Similarly we define

$$H_n^{\surd} := \varinjlim \left(\frac{1}{2^t}H_n, d\right)$$

and the GL_n-analogue is obtained.

THEOREM 2.5 ([1]): *Let* $m \in \mathbf{N}[\frac{1}{2}]$ *and* $m = 2^N - \sum_i^\ell 2^{r_i}$, *where* $r_1 < \cdots < r_\ell < N$. *Then* ${}_n\omega_m$ *can be written as a polynomial in terms of generators.*

$${}_n\omega_m = \sum_{\substack{2^{s_1}+\cdots+2^{s_\ell}=2^N \\ n \ge s_i - r_i \ge 0}} {}_n\omega_{2^{s_1}-2^{r_1}} \cdots {}_n\omega_{2^{s_\ell}-2^{r_\ell}}.$$

PROPOSITION 2.6:

(i) $H_n^{\surd} \cong \left(P_n^{\surd}\right)^{U_n}$, $n < \infty$.

(ii) $H_n^{\surd}$ *is the free root algebra generated by* $\{{}_1o, \ldots, {}_n o\}$, *where* ${}_io$ *is the image of* h_i *under the embedding of* H_n *in* $H_n^{\surd}$.

(iii) *Let* $n \in \mathbf{N}[\frac{1}{2}]$; *then* ${}_no_m$ *can be written as a polynomial in terms of generators.*

Proof: (iii) We suppose that m is an integer, otherwise we multiply by a suitable power of 2. Let $m = 2^N - \sum_i^\ell 2^{r_i}$; then $\alpha(m) = N - \ell - r_1$ and $\nu(m) = r_1$. Let $r = n - \alpha(m)$; then $R := r_1 + 1 - r$ is the biggest power of 2 such that $y_n^{2^R} | {}_no_m$

and ${}_no^{2^R}|{}_no_m$. Let ${}_no_m = {}_no^{2^R}f$; we compare the coefficients of $y_n^{2^R}$ in both sides: ${}_{n-1}\omega_{m-2^R} = ({}_{n-1}\omega_{2^{n-1}-1})^{2^R}f$. Now we use Arnon's theorem:

$$ {}_{n-1}\omega_{m-2^R} = \sum_{\substack{2^{s_0}+\cdots+2^{s_\ell}=2^N \\ n-1\geq s_i-r_i\geq 0}} {}_{n-1}\omega_{2^{s_0}-2^{r_0}} \cdots {}_{n-1}\,\omega_{2^{s_\ell}-2^{r_\ell}}. $$

Since it is known how to express a Dickson generator in terms of upper triangular generators, we only have to show that each summand in the last sum is divisible by $({}_{n-1}\omega_{2^{n-1}-1})^{2^R}$. For this we consider the indices of ω's. We require $s_i - r_i = n-1$ and $r_i = R+a$ for some positive integer a. This is equivalent to $s_i = n-1+R+a$ or $s_i = N-\ell-1+a$ and the last holds because $s_i \geq N-\ell$. ∎

For use later on, we consider the monomial $h^I = h_1^{i_1}\cdots h_n^{i_n} \in H_n^{\surd}$ so that I is a sequence of length n with entries from $\mathbf{N}[\frac{1}{2}]$. There is a unique least t in $\mathbf{N}$ with the property that $J = 2^tI$ is a sequence in $\mathbf{N}$. So each such monomial $h^I \in H_n^{\surd}$ uniquely determines the monomial $h^J \in (1/2^t)H_n$.

Let $\kappa_n\colon P_n \to P[n-1]$ be the degree-preserving algebra map defined on generators by the rule $\kappa_n(x_i) = x_i$ if $0 \leq i \leq n-1$ and $\kappa_n(x_n) = 0$. Later, we drop the subscript for convenience. Let us observe from the definitions that $\kappa_n(h_i) = h_i$ for $0 \leq i \leq n-1$, $\kappa_n(h_n) = 0$ while $\kappa_n(d_{i,n}) = d_{i,n-1}^2$ for $0 \leq i \leq n-1$, and $\kappa_n(d_{n,n}) = 0$.

LEMMA 2.7: $\kappa_n({}_n\omega_m) = {}_{n-1}\omega_m$.

The following diagram is easily induced since κ_n is compatible with d.

$$
\begin{array}{ccccccc}
\cdots \xrightarrow{\kappa} & P_{n+1}^{\surd} & \xrightarrow{\kappa} & P_n^{\surd} & \xrightarrow{\kappa} \cdots \\
& \uparrow & & \uparrow & \\
\cdots \xrightarrow{\kappa} & H_{n+1}^{\surd} & \xrightarrow{\kappa} & H_n^{\surd} & \xrightarrow{\kappa} \cdots \\
& \uparrow & & \uparrow & \\
\cdots \xrightarrow{\kappa} & D_{n+1}^{\surd} & \xrightarrow{\kappa} & D_n^{\surd} & \xrightarrow{\kappa} \cdots
\end{array}
$$

The system above induces the following inverse limits and maps:

$$ P\surd := \varprojlim(P_n^{\surd},\kappa), H\surd := \varprojlim(H_n^{\surd},\kappa) \text{ and } D\surd := \varprojlim(D_n^{\surd},\kappa); $$
$$ P\surd \hookleftarrow H\surd \hookleftarrow D\surd. $$

A typical element in one of these algebras can be described as follows. Let S be the set of sequences $I = (i_1, i_2, \ldots)$ with $i_k \in \mathbf{N}[\frac{1}{2}]$ and $i_k = 0$ for $k >> 0$.

Let $\mathcal{S}_k$ be the subset for which $i_j = 0$ when $j > k$. Given $I \in \mathcal{S}$ we write $g^I = \prod_k g_k^{i_k}$. Consider an expression of the form $[f = \sum_{I \in \mathcal{S}} a_I g^I]$. Write $\sup p(f) = \{I \in \mathcal{S} | a_I > 0\}$. Then f defines an element of one of these algebras iff $\sup p(f) \cap \mathcal{S}_k$ is finite for all k.

Remark 2.1: $H\surd$ is just the root algebra of $\varprojlim(H_n, \kappa)$. This is because $\varprojlim(H_n, \kappa)$ is an algebra on $\{h_i | i \geq 1\}$ of infinite series f such that $\pi_n(f) \in H_n$ without using the Frobenius map d: $H_n \to \frac{1}{2} H_{n+1}$.

Let $D^{\surd}_{fin}$ be the free root algebra generated by $\{{}_\infty\omega_{2^n-1} | n \geq 1\}$. Here ${}_\infty\omega_{2^n-1}$ is a Peterson polynomial where any number of variables is allowed. Arnon notes that for fixed length ${}_\infty\omega_{2^n-1}$ is finite (using the natural projections $D\surd \stackrel{\pi_n}{\to} D^{\surd}_n$).

Let ${}_nI_t$ be the ideal generated by $\{{}_n\omega_{2^t-1}, \ldots, {}_n\omega_{2^n-1}\}$ for $1 \leq t \leq n$. We consider $\{{}_nI_t | 1 \leq t \leq n\}$ a basis for open sets at 0 in the vector space $D^{\surd}_n$. We give $D\surd$ a topology such that all projections π_n are continuous. This topology is induced by the following metric on $D\surd$. Let $\mu(f)$ be the lowest number n such that $\pi_n(f) \neq 0$. Define

$$d(f, g) = \frac{1}{\mu(f - g)}.$$

A basis for open sets at 0 is given by the set of ideals

$$\{< \omega_{2^t-1}, \omega_{2^{t+1}-1}, \cdots\cdots > | 1 \leq t\}.$$

Let $f \in D\surd$ and define a sequence f_k in $D\surd$ as follows:

$$\sup p(f_k) = \sup p(f) \cap \mathcal{S}_k.$$

It is obvious that the sequence above converges to f and its elements belong to $D^{\surd}_{fin}$. Hence, $D\surd$ contains $D^{\surd}_{fin}$ as a dense subset. The same is true for the analogue of the U_n-invariants.

Note: Let A_μ be the inverse limit of $(\frac{1}{2^t}A, \mu)$. This vector space is considered as a topological space where a basis for open sets at 0 is given by $\{I_t | t > 0\}$. Here I_t is the kernel of the projection π_t: $A_\mu \to \frac{1}{2^t}A$, namely:

$$I_t = (0, \ldots, 0, A - A^{(2)}, A - A^{(2^2)}, \ldots)$$

and $A^{(2^k)}$ contains all elements with exponents divisible by 2^k. Each $(1/2^t)A$ is given the discrete topology. Suppose $\{\bar{x}_m = \{x_{m,n}\}\}$ is a Cauchy sequence; hence $\forall t\ \exists N_t$ such that $\bar{x}_m - \bar{x}_n \in I_t\ \forall m, n > N_t$. If we define $\bar{x} = \{x_t\}$ such that x_t is the summand of $\bar{x}_{N_t}$ which contains elements of sequences divisible by

2^t, $\bar{x}_m$ converges to $\bar{x}$ under this topology and A_μ is the completed Hopf algebra. Now, $D_n^{\surd}$ inherits a continuous action of A_μ which is compatible with the usual action of the Steenrod algebra on the Dickson algebra D_n, [1]. The same is true for $H_n^{\surd}$. This action extends on $P\surd$ and hence on $H\surd$ and $D\surd$. It is easy to show that this action is continuous.

A number of authors [7, 3, 6] have studied a coproduct

$$\Delta\colon D_n \to \bigoplus_{i+j=n} D_i \otimes D_j.$$

In particular, we have

$$\Delta(d_{n,n}) = \sum_{i+j=n} d_{i,i}^{2^j} \otimes d_{j,j}.$$

Hence, $D_{fin}^{\surd}$ acquires the structure of a Hopf algebra which is compatible with the structure above. Namely, one can take a formula describing the effect of D on certain special elements in D_n and copy it over to define a coproduct on $D_{fin}^{\surd}$ as the following diagram suggests:

$$\begin{array}{ccccccc} D_{fin}^{\surd} & \to & D_n^{\surd} & \to & \sum D_i^{\surd} \otimes D_j^{\surd} & \to & D_{fin}^{\surd} \otimes D_{fin}^{\surd} \\ \| & & \downarrow & & \downarrow & & \| \\ D_{fin}^{\surd} & \to & D_{n-1}^{\surd} & \to & \sum D_s^{\surd} \otimes D_t^{\surd} & \to & D_{fin}^{\surd} \otimes D_{fin}^{\surd} \end{array}$$

Here, the general formula is used:

$$\Delta_n(d_{n,n-i}) = \sum_{i+j=n} d_{k,k}^{2^{n-k}-2^j} d_{k,k-i+j}^{2^j} \otimes d_{n-k,n-k-j}, \quad [6].$$

We recall [8] that $A^* \cong \mathbf{F}_2[\xi_1, \xi_2, \ldots]$ where $|\xi_n| = 2^n - 1$ with coproduct

$$\Delta(\xi_n) = \sum_{i+j=n} \xi_i^{2^j} \otimes \xi_j.$$

THEOREM 2.8: *$\mathbf{F}_2[d_{n,n} | n \geq 1]$ and A^* are isomorphic as Hopf algebras.*

Remark 2.2: The last theorem implies that $D_{fin}^{\surd}$ contains a sub-Hopf algebra which is isomorphic to the dual Steenrod algebra.

3. The completed Dyer–Lashof algebra

We recall here for completeness the definition of the Dyer–Lashof together with an auxiliary algebra of some independent interest. Many readers will already be familiar with this material. An excellent reference is [3].

Let F be the free associative algebra on symbols $\{f^i \mid i \geq 0\}$, with $|f^i| = i$. Given a sequence $I = (i_1, \ldots, i_n)$ of non-negative integers we define the **length, degree** and **excess** of $f^I := f^{i_1} \cdots f^{i_n} \in F$ by $\ell(f^I) = n$, $|f^I| = i_1 + \cdots + i_n$ and $\operatorname{exc}(f^I) = i_1 - i_2 - \cdots - i_n$, respectively. We note that F is a Hopf algebra with coproduct defined on generators by the rule $\psi(f^i) = \sum_{j+k=i} f^j \otimes f^k$.

Let L be the two-sided ideal of F generated by the elements f^I of negative excess, and define W to be the quotient algebra F/L; we let e^I denote the image of f^I in W. We take W_m to be the subspace of W spanned by $\{e^I \| e^I| = m\}$. We note that $W_0 = \mathbf{F}_2[e^0, e^0e^0, \ldots]$ and we let $W[n]$ denote the subspace of W spanned by $\{e^I \mid |\ell(e^I) = n\}$ with $W[0] = \mathbf{F}_2$. We have $W = \bigoplus_{m \geq 0} W_m = \bigoplus_{n \geq 0} W[n]$. It is easy to see that ψ gives W the structure of a Hopf algebra.

We take L'' to be the two-sided ideal of W generated by the **Adem relations**: if $r > 2s$,

$$e^r e^s + \sum \binom{i-s-1}{2i-r} e^{r+s-i} e^i.$$

The **Dyer–Lashof algebra** R is defined to be the quotient algebra W modulo the ideal generated by the Adem relations W/L'. The image of e^I in R under the natural map π: $W \to R$ is denoted Q^I. An element Q^I (or I itself) is said to be **admissible** if $i_\ell \leq 2i_{\ell+1}$ for $1 \leq \ell \leq n-1$. The set $\{Q^I \mid I \text{ is admissible}\}$ is a $\mathbf{F}_2$-basis for R. R is a Hopf algebra under the coproduct induced by ψ, namely on generators by $\psi(Q^i) = \sum Q^{i-j} \otimes Q^j$, and if $R[n]$ denotes $\phi(W[n])$ then $R[n]$ is a connected subcoalgebra and $R = \bigoplus_{n \geq 0} R[n]$ as a coalgebra. The product in R sends $R[n] \otimes R[\ell]$ to $R[n+\ell]$ and the elements Q^i, $i \geq 0$ are all indecomposables.

THEOREM 3.1 ([2, 3, 7, 10]): *$W[n]^* \cong H_n$ and $R[n]^* \cong D_n$ as algebras over the Steenrod algebra. The first isomorphism sends $(e^I)^*$ to $h^{I'}$, where*

$$I' = (i_1 - \sum_2^n i_t, \ldots, i_{n-1} - i_n, i_n).$$

Following Arnon, we now consider the map μ: $F \to F$ defined on generators by the rule

$$\mu(f^{2i}) = f^i, \qquad \mu(f^{2i+1}) = 0.$$

The map μ can be extended to all of F by requiring that μ be a map of algebras and it is routine to check that μ is a map of Hopf algebras.

LEMMA 3.2: *The map μ induces Hopf algebra epimorphisms μ: $W \to W$ and μ: $R \to R$.*

LEMMA 3.3: *The map μ: $W[n] \to W[n]$ is dual under the isomorphism described above to the squaring map d: $H_n \to H_n$, and the same is true for μ: $R[n] \to R[n]$ and d: $D_n \to D_n$.*

Proof: It suffices to prove the assertion for μ: $W[n] \to W[n]$. The second assertion will follow from the commutativity of the diagram

$$\begin{array}{ccc} W[n] & \overset{\mu=d^*}{\to} & W[n] \\ \downarrow & & \downarrow \\ R[n] & \overset{\mu}{\to} & R[n] \end{array}$$

Let d^*: $H_n^* \cong W[n] \to H_n^* \cong W[n]$. Let $d^*(f^{I'})^* = d^* e^I$ (exponents are as in the theorem above). We evaluate the last element under f^J:

$$\begin{aligned} f^J(d^*(e^I)) &= df^J(e^I) = f^{2J}(e^I) = \begin{cases} 1 & \text{if } I = 2J' \\ 0 & \text{otherwise} \end{cases} \Rightarrow \\ d^* e^I &= \begin{cases} e^{I/2} & I/2 \in \mathbf{N} \times \cdots \times \mathbf{N}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$ ∎

Arnon's idea is to study a kind of "mapping telescope" of the function μ. In the first instance one needs a degree preserving map so let a map, still called μ: $\frac{1}{2}F \to F$, be defined where now $(\frac{1}{2}F)_n = F_{2n}$. Now consider $\frac{1}{2}F$ and F as algebras graded over $\mathbf{N}[\frac{1}{2}]$ with the latter algebra 0 in fractional degrees. Both algebras are still Hopf algebras but now μ is a degree-preserving map. Iterating the construction just given, obtain

$$\mu\colon \frac{1}{2^t}F \to \frac{1}{2^{t-1}}F$$

and consider the inverse limit

$$F_\mu = \varprojlim \left(\frac{1}{2^t}F, \mu \right).$$

Note that an arbitrary element of F_μ is an infinite sum of elements of the form

$$f(I/2^t) = \left((f^{I/2^t})_1, (f^{I/2^{t-1}})_2, \ldots, (f^I)_t, (f^{2I})_{t+1}, (f^{4I})_{t+2}, \ldots \right)$$

where I is a finite sequence of non-negative integers and we have written $(f^I)_t$ to indicate that $f^I \in \frac{1}{2^t}F$, $t \geq 0$. Hence, we allow exponents from the set $\mathbf{N}[\frac{1}{2}] \times \cdots \times \mathbf{N}[\frac{1}{2}]$ for any finite number of copies. So an arbitrary element $f \in F_\mu$ may be written as an infinite sum $\sum_\ell f(I_\ell)$. Note that the sequences I_ℓ may vary in length. Now, F_μ becomes a completed Hopf algebra where the coproduct is defined on $f(I)$ as $\sum_{J+K=I} f(J)\otimes f(K)$ (see [5]). Some care is required here. If one defines

$$F_\mu \bar{\otimes} F_\mu = \varprojlim \left(\frac{1}{2^t}F \otimes \frac{1}{2^r}F, \mu\otimes\mu\right)$$

we observe [5] that

$$F_\mu \bar{\otimes} F_\mu \cong \varprojlim \left(\frac{1}{2^t}F \otimes \frac{1}{2^t}F, \mu\otimes\mu\right).$$

The map μ induces an isomorphism of completed Hopf algebras μ: $\frac{1}{2}F_\mu \to F_\mu$ which maps $f(I,t)$ to $f(I,t+1)$. We define $F_\mu[n]$ in the obvious way and observe that $F_\mu[n] = \varprojlim(F[n],\mu)$ is a subcoalgebra of F_μ.

We observe that W_μ, $W_\mu[n]$, R_μ and $R_\mu[n]$ may be similarly defined and enjoy similar properties, since μ preserves Adem relations and negative excess. It is also true that $\mu(e^{2I}) = e^I$. We have $\Delta(e(I)) = \sum e(M)\otimes e(N)$.

It is known that A acts on R via Nishida relations and it follows in the sequel that A_μ also acts on R_μ via Nishida relations.

LEMMA 3.4: *A_μ acts on R_μ via Nishida relations.*

Proof: It suffices to show that the following diagram is commutative:

$$\begin{array}{ccccc} A & \bigotimes & R & \overset{Nishida}{\longrightarrow} & R \\ & \downarrow^{\mu\otimes\mu} & & \hookrightarrow & \downarrow^{\mu} \\ A & \bigotimes & R & \overset{Nishida}{\longrightarrow} & R \end{array}$$

This reduces to the statement

$$\binom{2s-2r-1}{2r-2i} \equiv \binom{s-r-1}{r-i} \bmod 2.$$

Note: Formulas for Adem relations and Nishida action are easily obtained.

Next, we discuss the action of R_μ on a completed homology of an infinite loop space.

The homology of an infinite loop space $H_*(QX)$ has a simple description over R, namely: it is an allowable R-Hopf algebra over a fixed homogeneous basis of $H_*(X)$ [3]. Let

$$(H_*(QX))^{\vee} := \varinjlim \left(\frac{1}{2^t}H_*(QX), d\right);$$

then R_μ acts on $(H_*(QX))^{\vee}$ as follows:

Definition 3.1: Let $\bar{u} \in (H_*(QX))^{\checkmark}$ and $Q(I) \in R_\mu$. Let $u_n \in (1/2^n)\,(H_*(QX))$ such that $u_n \notin \Im(d_{n-1})$ and $\bar{u} = \bar{u}_n$. For the exponent of $Q(I)$ there exists a non-negative integer t such that $2^t I$ is an integral sequence and t is the smallest with this property; let us label such a t by t_I. We define

$$Q(I)\bar{u} = \begin{cases} \overline{Q^{2^{n-t}I}u_n}, & \text{for } t \leq n, \\ 0, & \text{for } t > n. \end{cases}$$

PROPOSITION 3.5: *The completed Hopf algebra R_μ acts on $(H_*(QX, \mathbf{Z}_2))^{\checkmark}$.*

Proof: Let $Q \in R_\mu$; then $Q = \sum_I Q(I)$. Let $Q(n) := \sum_{t_I \leq n} Q(I)$. Here the summation is over all summands $Q(I)$ of Q such that $t_I \leq n$. The definition above is extended by $Q\bar{u} := \sum_I Q(I)\bar{u}$. The last sum is finite because $Q\bar{u} = Q(n)\bar{u}$ and hence the action is well defined. Here $\bar{u} \in (H_*(QX))^{\checkmark}$ and u_n is as in the definition above. ∎

4. The continuous dual of R_μ

In this section we consider the continuous dual of R_μ and its relation with the continuous dual of the completed Steenrod algebra A_μ.

First, we decompose R_μ (W_μ) with respect to length: Let

$$R_\mu[n] = \{Q = \sum_I Q(I) | I \text{ has length } n\};$$

then

$$R_\mu = \bigoplus_{n \geq 0} R_\mu[n] (W_\mu = \bigoplus_{n \geq 0} W_\mu[n]).$$

Definition 4.1: Let M be a $\mathbf{Z}_2$-graded topological vector space. Let M^* be the graded vector space of $\mathbf{Z}_2$-valued homogeneous continuous functionals on M.

Next, we define a scalar product between subspaces of R_μ and $D^{\checkmark}$. This scalar product will provide the main tool to investigate the dual space of $R_\mu[n]$ and $W_\mu[n]$.

We shall note that every element of $R_\mu[n]$ can be written uniquely in terms of $\{Q^{(1/2^t)I} | t \geq 0, I \text{ admissible}, e(I) \geq 0, \ell(I) = n\}$. The base for $D_n^{\checkmark}$ is the one which contains all monomials.

Definition 4.2: Let $e = \sum_J b_J e(J)$ be an element of $W_\mu[n]$ and $h = \sum_I a_I h^I$ an element of $H_n^{\checkmark}$. For each $J = (j_1, \ldots, j_n)$ define

$$J' = (j_1 - \sum_{s=2}^{n} j_s, \ldots, j_{n-1} - j_n, j_n) \quad \text{and} \quad b'_{J'} = b_J.$$

The scalar product between $W_\mu[n]$ and $H_n^{\surd}$ is defined to be given by

$$\langle e, h\rangle = \sum_I b'_I a_I.$$

Definition 4.3: Let $Q = \sum_J b_J Q(J)$ be an element of $R_\mu[n]$ and $d = \sum_I a_I d^I$ an element of $D_n^{\surd}$. Each integral sequence $2^{t_J} J = 2^{t_J}(j_1, \ldots, j_n)$ can be decomposed uniquely: $2^{t_J} J = \sum k_i I_i$ where $I_i = (2^{n-1} - 2^{i-1}, \ldots, 2^{n-i-1} - 1, 2^{n-i-2}, \ldots, 1)$ ([3]). Define $J' = 2^{-t_J}(k_1, \ldots, k_n)$ and $b'_{J'} = b_J$. The scalar product between $R_\mu[n]$ and $D_n^{\surd}$ is defined to be given by

$$\langle Q, d\rangle = \sum_I b'_I a_I.$$

LEMMA 4.1: *The scalar product is well defined and continuous.*

Proof: Let $h = \sum a_I h^I$ in $H_k^{\surd}$ and (e_n) be a convergent sequence in $W_\mu[n]$: $e_n = \sum b_{J_n} e^{J_n} \to e = \sum b_J e^J$. Thus, for any open I_m there exists N such that $(e_n - e) \in I_m$ for $n > N$. Let $t_0 = \max\{t_I | a_I \neq 0 \text{ in } h\}$. Let N be such that $(e_n - e) \in I_{t_0+1}$ for $n > N$. But this implies $\langle e_n - e, h\rangle = 0$. ∎

Let $f \in (W_\mu[n])^*$ or $(R_\mu[n])^*$; then the support of f is defined to contain all monomials of $W_\mu[n]$ or $R_\mu[n]$ such that their image under f is non-zero.

LEMMA 4.2: *Let $f \in (W_\mu[n])^*$ or $(R_\mu[n])^*$; then the support of f is finite.*

Proof: Since $f\colon W_\mu[n] \to \mathbf{Z}_2$ is continuous, there exists m such that $\forall e \in I_m$(a basic open set) with $|e| = |f|$ implies $f(e) = \dot{0}$. Let $x \in W_\mu[n]$ be a monomial such that $|x| = |f|$ and $x \notin I_m$. $x = e^J$ $(*)$ and $J = (\sum_{s=1}^n j_s, \sum_{s=2}^n j_s, \ldots, j_1)$, where $j_s \in \mathbf{N}[\frac{1}{2}]$. Hence, $|f| = \sum_{s=1}^n s j_s$. Since $x \notin I_m$, $2^t j_s \in \mathbf{N}$ for some $t \le m$ and $1 \le s \le n$. Let $k_{s,t} = 2^t j_s$; then the equation $2^t |f| = \sum_{s=1}^n s k_{s,t}$ has only a finite number of solutions. The last set of solutions defines the support of f.

Let $f \in (R_\mu[n])^*$ and $x = Q^J$ as above, $(*)$. Then there exists $t \le m$ such that $2^t J = \sum_i k_i I_{n,i}$ is a sequence in $\underbrace{\mathbf{N} \times \cdots \times N}_{n}$. As before $2^t |f| = \sum_i k_i (2^n - 2^i)$ has only a finite number of solutions. ∎

Now we are ready to proceed to the key theorem of this section.

THEOREM 4.3: *The maps $\varphi\colon H_k^{\surd} \to (W_\mu[n])^*$ given by $\varphi(h) = \langle -, h\rangle$ and $\psi\colon D_k^{\surd} \to (R_\mu[n])^*$ by $\psi(d) = \langle -, d\rangle$ are vector space isomorphisms.*

Proof: It is obvious that the maps defined above are monomorphisms. We must show that they are onto. So, let $f \in (W_\mu[n])^*$ with $|f| = d$. We define

$h(f) = \sum_{|J|=d} f(e^J)h^{J'}$. Since the support of f is finite, $h(f)$ is well defined and $\varphi(h(f))(e) = f(e)$. The last statement is true, since it holds for any monomial e^I. For the map ψ, the proof is identical. ∎

The next propositions are directly deduced from the theorem above.

PROPOSITION 4.4: *$(R_\mu)^* \cong \prod D_k^{\surd}$ and $(W_\mu)^* \cong \prod H_k^{\surd}$.*

PROPOSITION 4.5: *$R_\mu[n] \cong \left(D_k^{\surd}\right)^*$ and $W_\mu[n] \cong \left(D_k^{\surd}\right)^*$.*

Proof: This is a consequence of properties of direct limits.

THEOREM 4.6 ([5, theorem 3.7]): *$D_{fin}^{\surd}$ and A_μ^* are isomorphic as completed Hopf algebras.*

In his proof of this theorem, Hu'ng shows that the completion of A^* with respect to its squaring map is closely related to the graded hom-dual of Arnon's completion of the Steenrod algebra with respect to its halving map. Using the isomorphism between the Milnor basis and the top Dickson elements, one can show that the isomorphism above respects the underlined topologies.

Our last task is to complete the completed Dyer–Lashof algebra with respect to the map $\theta\colon F[n] \to F[n+1]$ defined by the rule $\theta(f^I) = f^{(2I,0)}$.

LEMMA 4.7: *θ is a map of coalgebras and also induces a coalgebra map from $W[n]$ to $W[n+1]$ and $R[n]$ to $R[n+1]$.*

Proof: The following diagram is commutative:

$$\begin{array}{ccc} W[n] & \xrightarrow{\theta} & W[n+1] \\ {\scriptstyle\Delta}\downarrow & & {\scriptstyle\Delta}\downarrow \\ W[n]\otimes W[n] & \xrightarrow{\theta\otimes\theta} & W[n+1]\otimes W[n+1] \end{array}$$

Since θ is compatible with excess and Adem relations, the second assertion follows. ∎

It is just a diagram chasing to see that θ and μ are compatible in W and R. It follows therefore that we may take a direct limit of the system

$$R_\mu[1] \xrightarrow{\theta} \cdots \xrightarrow{\theta} R_\mu[n] \xrightarrow{\theta} R_\mu[n+1] \xrightarrow{\theta} \cdots$$

to obtain a coalgebra $\widehat{R_\mu}$.

PROPOSITION 4.8: *The map $\theta^*\colon (R[n+1]_\mu)^* \to (R[n]_\mu)^*$ agrees with the map induced by $\kappa\colon D_{n+1}^{\surd} \to D_n^{\surd}$ under the isomorphism of Theorem 3.1.*

The last proposition implies the following theorem.

THEOREM 4.9: *$\left(\widehat{W_\mu}\right)^* \cong H\surd$ and $\left(\widehat{R_\mu}\right)^* \cong D\surd$ as algebras.*

Combining the last theorem with Theorem 4.6, the next corollary is obtained.

COROLLARY 4.10: *$(A_\mu)^*$ has an isomorphic dense image in $\left(\widehat{R_\mu}\right)^*$.*

References

[1] D. Arnon, *Generalized Dickson invariants*, Ph.D. Thesis, MIT, 1994.

[2] H. E. A. Campbell, *Upper triangular invariants*, Canadian Mathematical Bulletin **28** (1985), 243–248.

[3] F. Cohen, T. Lada and J. P. May, *The homology of iterated loop spaces*, Lecture Notes in Mathematics **533**, Springer-Verlag, Berlin, 1975.

[4] L. E. Dickson, *A fundamendal system of invariants of the general modular linear group with a solution of the form problem*, Transactions of the American Mathematical Society **12** (1911), 75–98.

[5] I. Llerena and N. H. V. Hu'ng, *The complete Steenrod algebra and the generalized Dickson algebra*, in *Algebraic Topology, New trends in localization and periodicity*, Progress in Mathematics 136, Birkhäuser, Boston, 1996.

[6] Nondas E. Kechagias, *Extended Dyer–Lashof algebras and modular coinvariants*, Manuscripta Mathematica **84** (1994), 261–290.

[7] I. Madsen, *On the action of the Dyer–Lashof algebra in $H_*(B)$*, Pacific Journal of Mathematics **60** (1975), 235–275.

[8] J. Milnor, *The Steenrod algebra and its dual*, Annals of Mathematics **67** (1958), 150–171.

[9] H. Mui, *Modular invariant theory and the cohomology algebras of the symmetric groups*, Journal of the Faculty of Science of the University of Tokyo. Section IA (1975), 319–369.

[10] H. Mui, *Homology operations derived from modular coinvariants*, Lecture Notes in Mathematics **1172**, Springer-Verlag, Berlin, 1985, pp. 85–115.

[11] C. W. Wilkerson, *A primer on Dickson invariants*, Contemporary Mathematics **19** (1983), 421–434.